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The solution of a torsion problem by finite Mellin transform techniques

J. TWEED

Department of Mathematics, University of Glasgow, University Garden, Glasgow, U.K.

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SUMMARY
In this paper the author makes use of a recent development in the theory of finite Mellin transforms to find formulae for the stress intensity factors and crack formation energy of a radial system of edge cracks in a circular elastic cylinder under torsion.

1. Introduction

In a recent paper [1] the author and a colleague considered the problem of determining the stress intensity factors and the crack energy of a radial system of edge cracks in a circular elastic cylinder under torsion. Since the method used in that paper is rather involved we now wish to show that by utilising the properties of the finite Mellin transform $M_R[f(r); s]$, which is defined by the equation

$$M_R[f(r); s] = \int_0^R [r^{s-1} + R^{2s} r^{-s-1}] f(r) dr , \tag{1.1}$$

we can find the solution of this problem in a much simpler fashion. This transform was first introduced by D. Naylor in his paper [2] and recently [3] the present author has proved the following form of the inversion theorem.

Theorem. *Let $y^{c-1} f(y) \in L(0, R)$ for every real number c such that $|c| < \sigma$ and let $f(y)$ be of bounded variation in the neighbourhood of the point $y = x \in (0, R)$. Let*

$$\bar{f}(s) = \int_0^R [x^{s-1} + R^{2s} x^{-s-1}] f(x) dx \quad (s = c + ib)$$

Then

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s) x^{-s} ds , \quad |c| < \sigma .$$

We shall assume that, in cylindrical coordinates (r, ϑ, z), the cylinder is defined by the relations $0 \leqq r \leqq b$, $0 \leqq \vartheta \leqq 2\pi$, $0 \leqq z \leqq L$ and the cracks by $0 < bc \leqq r \leqq b$, $\vartheta = 2k\pi/n$, $k = 0, 1, 2, \ldots, n-1$. It is supposed further, that the end $z = 0$ is fixed in the $r\vartheta$-plane and that the other end is acted upon by a couple whose moment **T** lies along the z-axis and which produces in the cylinder a twist of α per unit length. With these assumptions it can be shown (see [4]) that the displacements may be written in the form

$$u = 0 , \quad v = \alpha r z , \quad w = \alpha \varphi(r, \vartheta) \tag{1.2}$$

and that the corresponding stresses are given by

$$\sigma_{rr} = \sigma_{zz} = \sigma_{\vartheta\vartheta} = \sigma_{r\vartheta} = 0 ,$$

$$\sigma_{rz} = \mu\alpha \frac{\partial \varphi}{\partial r} \quad \text{and} \quad \sigma_{z\vartheta} = \frac{\mu\alpha}{r} \left[\frac{\partial \varphi}{\partial \vartheta} + r^2 \right] , \tag{1.3}$$

where μ is the shear modulus and $\varphi(r, \vartheta)$ is a solution of the partial differential equation

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \vartheta^2} = 0 . \tag{1.4}$$

Furthermore, the torsional rigidity D is given by

$$D = \mu \iint_A \left[\frac{\partial \varphi}{\partial \vartheta} + r^2\right] r\, dr\, d\vartheta , \tag{1.5}$$

where A is a cross-section of the cylinder, and is related to α and T through the equation

$$\alpha = T/D . \tag{1.6}$$

2. The reduction of the problem to a pair of dual equations

By using a symmetry argument and making use of the fact that the crack surfaces and the lateral surfaces of the cylinder are traction free, it is not difficult to show that the problem may be reduced to that of finding a function $\varphi(r, \vartheta)$ which satisfies the equation (1.4) in the sector $0<r<b$, $0<\vartheta<\pi/n$ and which has the following properties.

(1) φ, $\dfrac{\partial \varphi}{\partial r}$ and $\dfrac{\partial \varphi}{\partial \vartheta}$ are all finite at $r=0$,

(2) $\varphi(r, \pi/n)=0$, $\quad 0 \leqq r \leqq b$,

(3) $\dfrac{\partial \varphi}{\partial r}(b, \vartheta)=0$, $\quad 0 \leqq \vartheta \leqq \pi/n$,

(4) $\varphi(r, 0)=0$, $\quad 0 \leqq r \leqq bc$

and

(5) $\dfrac{\partial \varphi}{\partial \vartheta}(r, 0) = -r^2$, $\quad bc<r<b$.

In order to find such a function we apply the transform M_b to the equation (1.4) and make use of the result

$$M_b\left[\left(r\frac{\partial}{\partial r}\right)^2 f(r); s\right] = s^2 M_b[f(r); s] + 2b^{s+1} f'(b) . \tag{2.1}$$

This yields the equation

$$\frac{d^2 \bar{\varphi}}{d\vartheta^2}(s, \vartheta) + s^2 \bar{\varphi}(s, \vartheta) = 0 , \tag{2.2}$$

where we have written

$$\bar{\varphi}(s, \vartheta) = M_b[\varphi(r, \vartheta); r \to s] . \tag{2.3}$$

Since condition (2) implies that $\bar{\varphi}(s, \pi/n)=0$, we find that

$$\bar{\varphi}(s, \vartheta) = \frac{A(s) \sin s(\vartheta - \pi/n)}{s \sin \pi s/n}$$

and hence that

$$\varphi(r, \vartheta) = M_b^{-1}\left[\frac{A(s) \sin s(\vartheta - \pi/n)}{s \sin \pi s/n}; s \to r\right], \tag{2.4}$$

where M_b^{-1} is the inverse of the transform M_b. On applying conditions (4) and (5) we now discover that $A(s)$ must satisfy the dual equations

$$\begin{aligned} &M_b^{-1}[s^{-1} A(s); r] = 0 , && 0 \leqq r \leqq bc \\ &M_b^{-1}[A(s) \cot \pi s/n; r] = -r^2 , && bc < r \leqq b . \end{aligned} \tag{2.5}$$

It has recently been shown [2] that the solution of these equations may be written in the form

$$A(s) = \int_{bc}^{b} t^{n-1} p(t^n)[b^{2s}t^{-s} - t^s]\,dt \tag{2.6}$$

where $p(t^n)$ is given by the formula

$$p(t^n) = \frac{-n}{\pi[(b^n - c^n t^n)(t^n - b^n c^n)]^{\frac{1}{2}}} \times \int_{bc}^{b} [(b^n - c^n x^n)(x^n - b^n c^n)]^{\frac{1}{2}} \left\{ \frac{1}{x^n - t^n} + \frac{b^n}{b^{2n} - x^n t^n} \right\} x\,dx\,. \tag{2.7}$$

Furthermore, it is also shown that

$$M_b^{-1}[s^{-1}A(s);\, r] = \int_{bc}^{r} t^{n-1} p(t^n)\,dt\,, \quad bc < r < b\,. \tag{2.8}$$

Now, from (1.5) we see that

$$D/\mu = \tfrac{1}{2}\pi b^4 - 2n \int_{bc}^{b} \varphi(r, 0)\, r\,dr$$

and hence on substituting from (2.4), (2.7) and (2.8) into this formula we find that

$$D/\mu = \frac{b^4}{\pi}\left[\frac{\pi^2}{2} - I_n\right], \tag{2.9}$$

where

$$I_n = n^2 \int_c^1 \frac{t^{n-1}(1-t^2)}{\Delta(t^n)}\,dt \int_c^1 x\,\Delta(x^n) \left\{ \frac{1}{x^n - t^n} + \frac{1}{1 - x^n t^n} \right\} dx \tag{2.10}$$

and

$$\Delta(t^n) = [(c^{-n} - t^n)(t^n - c^n)]^{\frac{1}{2}}\,. \tag{2.11}$$

3. The stress intensity factor and the crack energy.

In this section we calculate two important quantities which are of interest to workers in fracture mechanics. The first is the stress intensity factor K which is defined by the equation

$$K = \lim_{r \to bc+} [2(r - bc)]^{\frac{1}{2}} \mu \frac{\partial w}{\partial r}(r, 0)\,, \tag{3.1}$$

and the second is the crack energy W which is given by

$$W = n \int_{bc}^{b} \sigma_{z\vartheta}^{(1)}(r, 0)\, w(r, 0)\,dr\,, \tag{3.2}$$

where $\sigma_{z\vartheta}^{(1)}(r, 0) = \mu\alpha r$ is the shear stress on $\vartheta = 0$ in the absence of the cracks. From (1.2), (2.4) and (2.8) we see that

$$w(r, 0) = -\alpha \int_{bc}^{r} t^{n-1} p(t^n)\,dt\,, \quad bc < r < b \tag{3.3}$$

and hence that

$$\frac{\partial w}{\partial r}(r, 0) = -\alpha r^{n-1} p(r^n)\,, \quad bc < r < b\,. \tag{3.4}$$

It follows that

$$K = -\alpha\mu (bc)^{n-1} \lim_{r \to bc+} [2(r - bc)]^{\frac{1}{2}} p(r^n)\,. \tag{3.5}$$

On substituting from (2.7) into this equation and taking the limit we now see that

$$K = \frac{\mu\alpha b^{\frac{3}{2}}}{\pi}\left[\frac{2n(1-c^n)}{c(1+c^n)}\right]^{\frac{1}{2}} \int_c^1 \frac{y(1+y^n)}{\Delta(y^n)}\, dy\,, \tag{3.6}$$

where $\Delta(y^n)$ is given by (2.11). But $\alpha = T/D$ and therefore, on making use of (2.9), we find that

$$K = \frac{T}{b^{\frac{5}{2}}(\frac{1}{2}\pi^2 - I_n)}\left[\frac{2n(1-c^n)}{c(1+c^n)}\right]^{\frac{1}{2}} \int_c^1 \frac{y(1+y^n)}{\Delta(y^n)}\, dy\,, \tag{3.8}$$

where I_n is given by (2.10). Similarly, by substituting from (3.3) into (3.2), it is not difficult to show that W is given by the formula

$$W = \frac{\pi T^2}{2\mu b^4}\,\frac{I_n}{(\frac{1}{2}\pi^2 - I_n)^2}\,. \tag{3.9}$$

The results obtained here agree with those given in [1] where the cases $n = 1$ and 2 are considered in detail and numerical results given.

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